25 PROBLEMS & SOLUTIONS ON DIFFERENTIATION_1st ORDER

1. Find $\frac{dy}{dx}$ where $y=\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\cot^{-1}\left(x^2+5x+7\right)$.

[SOL : $y=\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\cot^{-1}\left(x^2+5x+7\right)$

$=\tan^{-1}\frac{(x+1)-x}{1+x(x+1)}+\tan^{-1}\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}+\frac{\pi}{2}-\tan^{-1}\frac{(x+3)-(x+2)}{x^2+5x+7}$

$=\tan^{-1}(x+1)-\tan^{-1}x+\tan^{-1}(x+2)-\tan^{-1}(x+1)+\tan^{-1}(x+3)-\tan^{-1}(x+2)$

$=\tan^{-1}(x+3)-\tan^{-1}x$

Now, differentiating both sides w.r. to x, we get,

$\frac{dy}{dx}=\frac{d}{d(x+3)}\tan^{-1}(x+3)\times\frac{d(x+3)}{dx}-\frac{1}{1+x^2}$

$=\frac{1}{1+(x+3)^2}\times(1+0)-\frac{1}{1+x^2}=\frac{1}{1+x^2+6x+9}-\frac{1}{1+x^2}=\frac{1+x^2-x^2-6x-10}{(x^2+6x+10)(1+x^2)}$

$=\frac{-6x-9}{(x^2+6x+10)(1+x^2)}$

$\therefore \frac{dy}{dx}=\frac{-6x-9}{(x^2+6x+10)(1+x^2)}$]

2. If $y=\tan^{-1}\frac{3a^2x-x^3}{a\left(a^2-3x^2\right)}$ then prove that $\frac{dy}{dx}=\frac{3a}{a^2+x^2}$.

[SOL : Given, $y=\tan^{-1}\frac{3a^2x-x^3}{a\left(a^2-3x^2\right)}=\tan^{-1}\left[\frac{3\left(\frac{x}{a}\right)-\left(\frac{x}{a}\right)^3}{1-3\left(\frac{x}{a}\right)^2}\right]$

Let, x = a.tanθ $\Rightarrow \frac{dx}{d\theta}=a.\sec^2\theta=a(1+\tan^2\theta)=a(1+\frac{x^2}{a^2})=\frac{a^2+x^2}{a}$

Then, $y=\tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]=\tan^{-1}[\tan 3\theta]=3\theta$

$y=3\theta \quad \Rightarrow \frac{dy}{d\theta}=3$ Hence, $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3}{\frac{a^2+x^2}{a}}=\frac{3a}{a^2+x^2}$ Proved]

3. If $y=\tan^{-1}\left(\frac{\tan x+\sec x-1}{\tan x-\sec x+1}\right)$ then prove that $\frac{dy}{dx}=\frac{1}{2}$.

[SOL : Given, $y=\tan^{-1}\left(\frac{\tan x+\sec x-1}{\tan x-\sec x+1}\right)=\tan^{-1}\left(\frac{\tan x+\sec x-(\sec^2 x-\tan^2 x)}{\tan x-\sec x+1}\right)$

$\therefore y=\tan^{-1}\left(\frac{(\tan x+\sec x)(\tan x-\sec x+1)}{(\tan x-\sec x+1)}\right)=\tan^{-1}\left(\frac{\sin x+1}{\cos x}\right)$

$=\tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)}\right\}=\tan^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\}=\tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\}$

$$= \frac{ab}{(a^2+b^2)} \frac{\not{2}\left(a^2\cos^2\frac{\theta}{2} - b^2\sin^2\frac{\theta}{2}\right)}{\not{2}\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} = \frac{ab}{(a^2+b^2)}\left(a^2\cot\frac{\theta}{2} - b^2\tan\frac{\theta}{2}\right)$$

Hence, $\frac{dy}{dz} = \frac{ab}{a^2+b^2}\left(a^2\cot\frac{\theta}{2} - b^2\tan\frac{\theta}{2}\right)$ (proved)]

25. If $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \cdots\cdots\infty}}}}$ then show that $\frac{dy}{dx} = \frac{y^2 - x}{2y^3 - 2xy - 1}$.

[SOL : Given $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \cdots\cdots\infty}}}}$ $\Rightarrow y^2 = x + \sqrt{y + \sqrt{x + \sqrt{y + \cdots\cdots\infty}}}$

$\Rightarrow (y^2 - x)^2 = y + \sqrt{x + \sqrt{y + \cdots\cdots\infty}}$ $\Rightarrow (y^2 - x)^2 = 2y$

$\Rightarrow 2(y^2 - x)\left(2y\frac{dy}{dx} - 1\right) = 2\frac{dy}{dx}$ [Differentiating both sides w.r.to x]

$\Rightarrow 4y\frac{dy}{dx}(y^2 - x) - 2(y^2 - x) = 2\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(2y^3 - 2xy - 1) = 2(y^2 - x)$

Hence, $\frac{dy}{dx} = \frac{y^2 - x}{2y^3 - 2xy - 1}$ (proved)]

"The essence of Mathematics, lies in its freedom" –
G. CANTOR

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